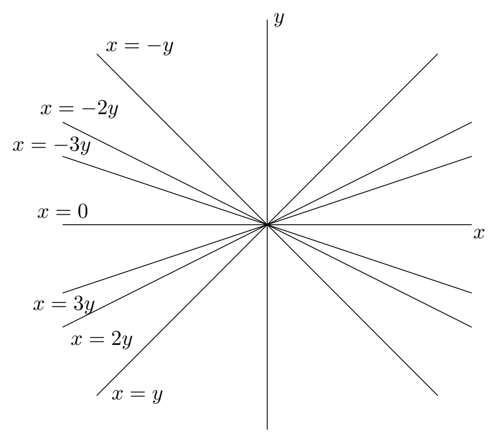
Solutions HW 2

2.1.10

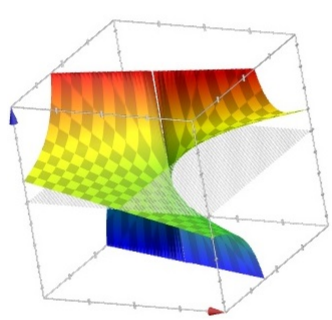
a) The level sets are circles with radius $\sqrt{c-1}$ for $c \geq 1$ and empty otherwise.
b) The level sets are circles with radius $\sqrt{1-c}$ for $c \leq 1$ and empty otherwise.
c) For each $c \in \mathbb{R}$, the level sets are between one and three lines parallel to the y-axis of the form (α, t) with $t \in \mathbb{R}$ and α such that $\alpha^3 - \alpha = c$. To see this, let α be any root of $x^3 - x = c$, then $f(\alpha, y) = 0$ for all values of y. Conversely if $f(x, y) = 0$, x has to be a root of the previous polynomial.

2.1.18

If $x/y = c$ we have that $x = cy$, then we have the following level curves:



And by putting them together we get the following 3D sketch (obtained with Google):



2.1.33

For all $y \in \mathbb{R}$ we have that the section in the xz plane is the parabola $z = x^2$. Hence the surface is a cylinder of parabolas along the y axis.

2.1.34

For all $x \in \mathbb{R}$ we have that the section in the zy plane is the circle of radius 2 centered at $(0,0)$, hence the surface is a cylinder along the x axis.

2.1.38

The resulting figure is an ellipsoid (like a deformed ball) which cuts the x axis at $(3,0,0)$ and $(-3,0,0)$, cuts the y axis at $(0, 2\sqrt{3}, 0)$ and $(0, -2\sqrt{3}, 0)$, and cuts the z axis at $(0,0,3)$ and $(0,0,-3)$

2.2.6

a) $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \dfrac{0 \cdot y^3}{0^2 + y^6} = \lim_{y\to 0} \dfrac{0}{y^6} = 0$

b) $\lim_{y\to 0} f(y^3,y) = \lim_{y\to 0} \dfrac{y^3 \cdot y^3}{(y^3)^2 + y^6} = \lim_{y\to 0} \dfrac{y^6}{2y^6} = \lim_{y\to 0} \dfrac{1}{2} = \dfrac{1}{2}$

c) If f is continuous at $(0,0)$, all the limits through paths that go to $(0,0)$ should be equal, and since we have already found two different limits from a) and b), we deduce that f is not continuous at $(0,0)$.

2.2.8

a) Since $(x+y)^2 - (x-y)^2 = 4xy$ we have that

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2 - (x-y)^2}{xy} = \lim_{(x,y)\to(0,0)} \frac{4xy}{xy} = \lim_{(x,y)\to(0,0)} 4 = 4$$

b) The Taylor series of $\sin t$ at $t = 0$ shows that $\sin t = t + r_1(t)$, where the residue $r_1(t)$, which is formed by powers of 2 and greater, satisfies that

$$\lim_{t\to 0} \frac{r_1(t)}{t} = 0$$

Hence $\lim_{t\to 0} \dfrac{\sin t}{t} = \lim_{t\to 0} \dfrac{t + r_1(t)}{t} = 1 + 0 = 1.$

Then we have that

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = \left(\lim_{(x,y)\to(0,0)} x\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy}\right) = 0 \cdot 1 = 0$$

c) In polar coordinates, when substituting $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{r\to 0} \frac{(r\cos\theta)^3 - (r\sin\theta)^3}{(r\cos\theta)^2 + (r\sin\theta)^2} = \lim_{r\to 0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}$$

Which can be reduced to $\lim_{r\to 0} r(\cos^3\theta - \sin^3\theta)$, and this last limit is 0 since it is the product of a limit 0 function with a bounded function.

2.2.10

a) Since the function is continuous we can just evaluate at $(0,0)$:

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x+1} = \frac{e^0}{0+1} = 1$$

b) Taking the path $x = 0$ we have that

$$\lim_{y\to 0} \frac{\cos 0 - 1 - (0^2/2)}{0^4 + y^4} = \lim_{(x,y)\to(0,0)} \frac{0}{y^4} = 0$$

Now taking $y = 0$ we have to take the limit at 0 of $f(x) = \dfrac{\cos x - 1 - (x^2/2)}{x^4}$. Applying the Taylor series of $\cos x$ at 0 and using the same reasoning as in 2.2.8 b) [after x^4 the residue divided by x^4 goes to 0] we have that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1 - (x^2/2) + (x^4/24) - 1 - (x^2/2)}{x^4}$$

Which if simplified is equal to

$$\lim_{x\to 0} \frac{-x^2 + x^4/24}{x^4} = \lim_{x\to 0} \frac{-1 + x^2/24}{x^2} = -\infty$$

Hence we have found two different values for paths that approach the desired limit point, hence the limit doesn't exist.

c) Let $f(x,y) = \dfrac{(x-y)^2}{x^2+y^2}$, then approaching 0 by $x = y$ we have:

$$\lim_{y\to 0} f(y,y) = \lim_{y\to 0} \frac{(y-y)^2}{y^2+y^2} = \lim_{y\to 0} \frac{0}{2y^2} = 0$$

While taking the path $x = 0$ we have

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{(0-y)^2}{0^2+y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = \lim_{y\to 0} 1 = 1$$

And again we have found two different possibilities for the limit, which tells us that it doesn't exist.

2.2.23

This problem is just an application of Theorem 4 (page 98), which tells us that addition and product of continuous functions is continuous.